NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 2 Term 1 2016

Name: ________________ **Mathematics Class:** 12MZ__

Student Number: ______________

Time Allowed: **55 minutes + 2 minutes reading time**

Available Marks: **39**

Instructions:

- Questions are not of equal value.
- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Question	1-3	4-5	6ab	6c	6d	7a	7bc	7d	Total
E4	/3			/3		/3	/11		/20
E6		/2	/9		/4				/15
E9						/1		/3	/4
									/39

Section I

5 marks
Attempt Questions 1- 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1- 5

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1 The polynomial $x^3 - 2x^2 - 15x + 36$ has a double root at $x = \alpha$.

What is the value of α?

(A) $-\dfrac{5}{3}$

(B) -3

(C) 3

(D) $\dfrac{5}{3}$

2 $(2i+1)$ is a root of the equation $x^3 - 4x^2 + 9x - 10 = 0$.
Which of the following is another root?

(A) 5

(B) 10

(C) $2i - 1$

(D) 2

3 Given that $\dfrac{x+1}{x^2-4} \equiv \dfrac{A}{x+2} + \dfrac{B}{x-2}$, what are the values of A and B?

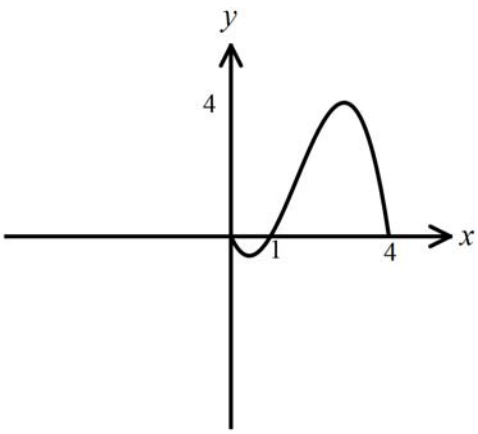
(A) $A = \dfrac{1}{4},\ B = \dfrac{3}{4}$

(B) $A = -\dfrac{1}{4},\ B = \dfrac{3}{4}$

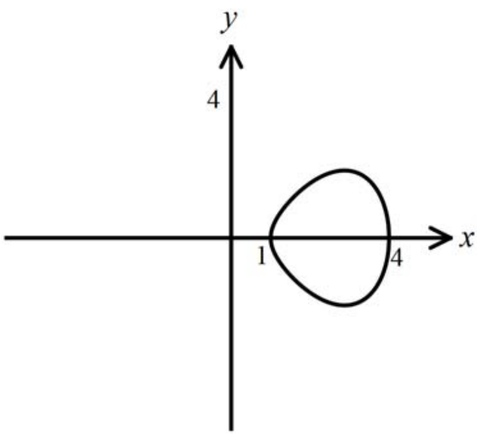
(C) $A = \dfrac{1}{4},\ B = -\dfrac{3}{4}$

(D) $A = -\dfrac{1}{4},\ B = -\dfrac{3}{4}$

4 The graph of $y = f(x)$ is shown.



A second graph is obtained from the function $y = f(x)$



Which equation best represents the second graph?

(A) $y^2 = |f(x)|$

(B) $y^2 = f(x)$

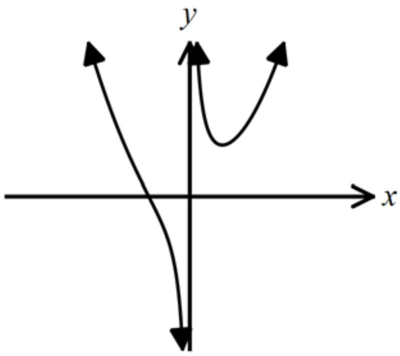
(C) $|y| = f(x)$

(D) $y = \sqrt{f(x)}$

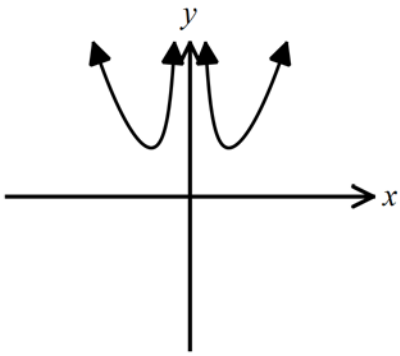
5 Let $f(x)=\dfrac{x^k+a}{x}$, where k and a are real constants.

Given that k is an odd integer greater than 1 and $a<0$, which of the following could be the graph of $y=f(x)$?

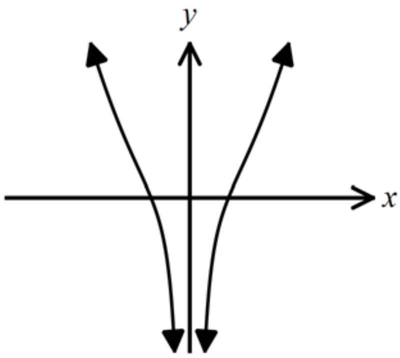
(A)



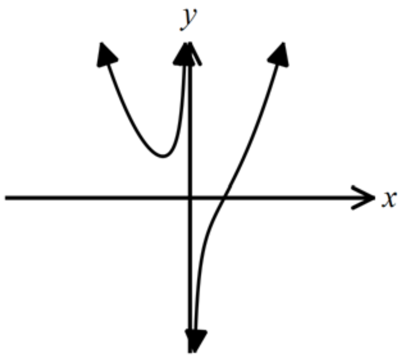
(B)



(C)



(D)



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Section II

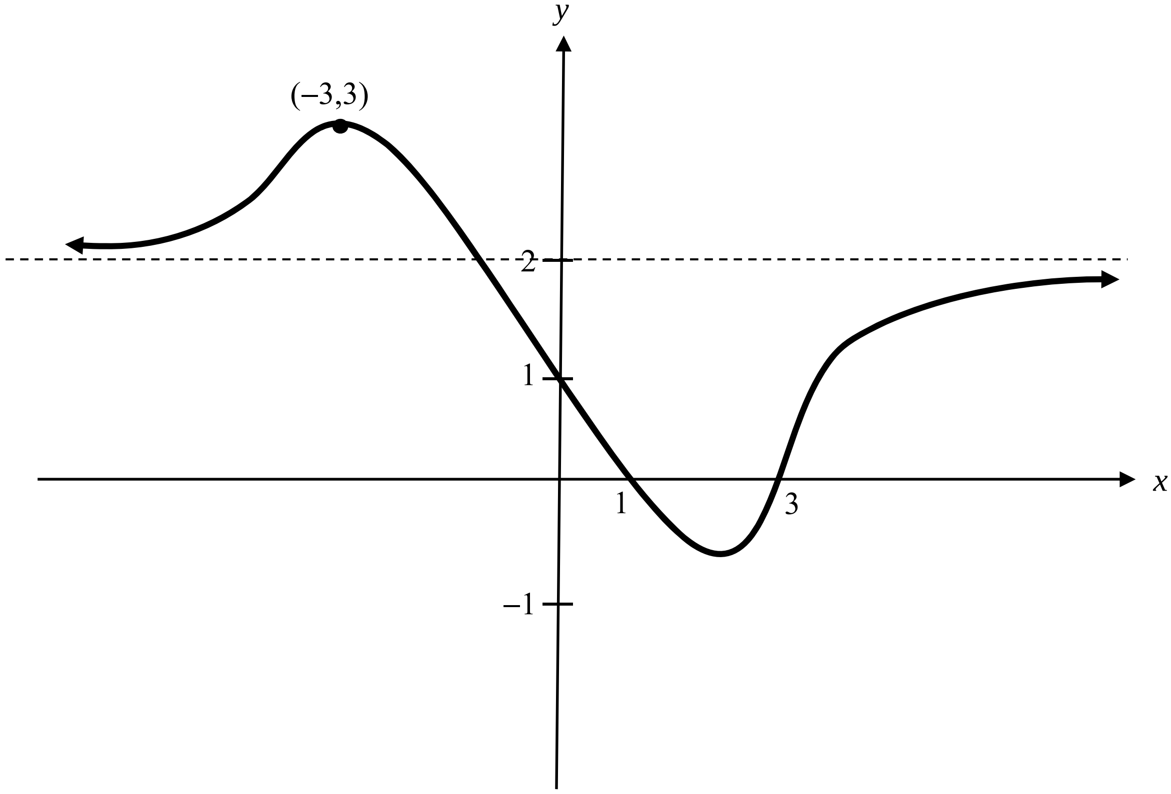
34 marks
Attempt Questions 6- 7
Allow about 47 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6 – 7, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (Use a SEPARATE writing booklet) **16 marks**

(a) Consider the graph of the function $y = f(x)$ shown below. There is a horizontal asymptote at $y = 2$.



On the response sheets provided, sketch neat graphs of the following graphs showing all important features. If any portion of the original curve is part of your answer, you must clearly indicate this.

(i) $y = f\left(\dfrac{x}{2}\right)$ **2**

(ii) $y = f(|x|)$ **2**

(iii) $y = \log_2 f(x)$ **2**

(b) If $g(x)=\begin{cases} f(x) & \text{if } 0\le x\le 3 \\ f(6-x) & \text{if } 3<x\le 6 \end{cases}$ where $f(x)$ is defined as in part (a).

(i) Neatly sketch $y=g(x)$. **2**

(ii) Solve $g(x)>0$. **1**

(c) Consider the polynomial $P(x)=x^3-x^2-21x+45$ with roots α, β and γ.

(i) Find the monic polynomial with roots $\alpha-3,\ \beta-3,\ \gamma-3$. **2**

(ii) Hence solve $P(x)=0$. **1**

(d) Consider the curve with equation $x^3+y^3-3xy=48$.

(i) Show that $\dfrac{dy}{dx}=\dfrac{y-x^2}{y^2-x}$. **2**

(ii) Find the $x-$coordinates of the stationary points of the curve. **2**

End of Question 6

Question 7 (Use a SEPARATE writing booklet) **18 marks**

(a) The polynomial $P(x) = x^4 - 8x^3 + 24x^2 - 32x + 20$ has $1+i$ as a zero.

(i) Express $P(x)$ as a product of two real quadratic factors. **3**

(ii) Explain briefly why $P(x) > 0$ for all real values of x. **1**

(b) The polynomial $P(x)$ is given by $P(x) = x^3 + ax^2 + bx + c$ where a, b and c are real. The equation $P(x) = 0$ has roots α, β and γ. S_n is defined by $S_n = \alpha^n + \beta^n + \gamma^n$ for $n = 1, 2, 3, \ldots$, and it is given that $S_1 = S_3 = 3$ and $S_2 = 7$.

(i) Show that $a = -3$ and $b = 1$. **2**

(ii) Find the value of c. **2**

(c) (i) Use De Moivre's Theorem to show $(\cot\theta + i)^n + (\cot\theta - i)^n = \dfrac{2\cos n\theta}{\sin^n\theta}$. **2**

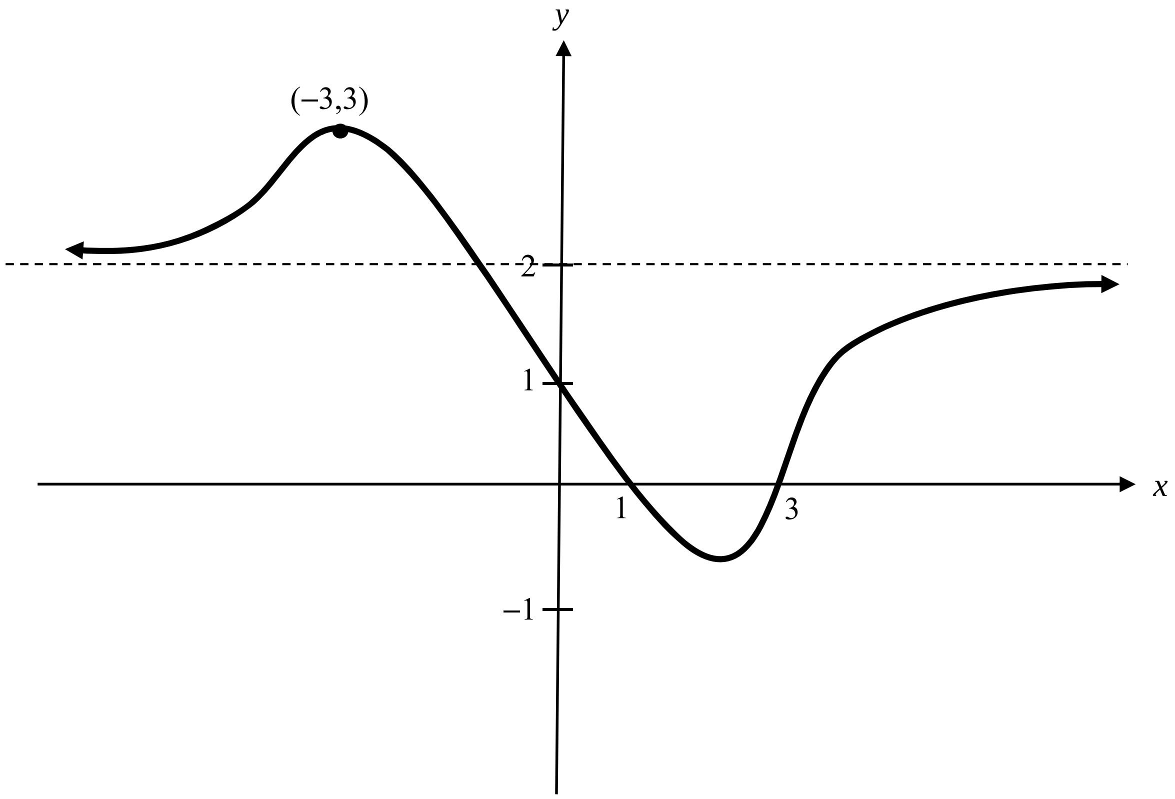
(ii) Show that the equation $(x+i)^5 + (x-i)^5 = 0$ has roots $0, \pm\cot\dfrac{\pi}{10}, \pm\cot\dfrac{3\pi}{10}$. **3**

(iii) Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots $\pm\cot\dfrac{\pi}{10}, \pm\cot\dfrac{3\pi}{10}$. **2**

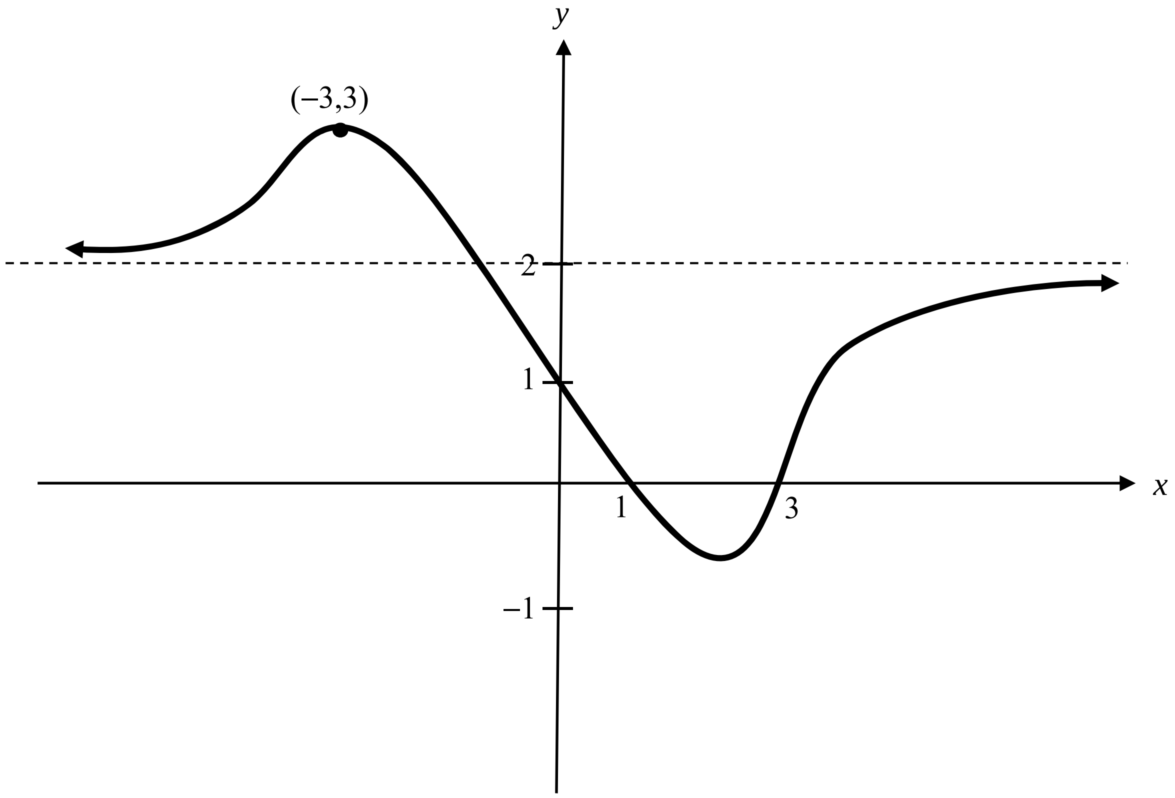
(d) Consider the function $f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}$. **3**
Find the maximum value of $f(x)$ using a graphical method.
No credit will be awarded for using calculus.

End of Test

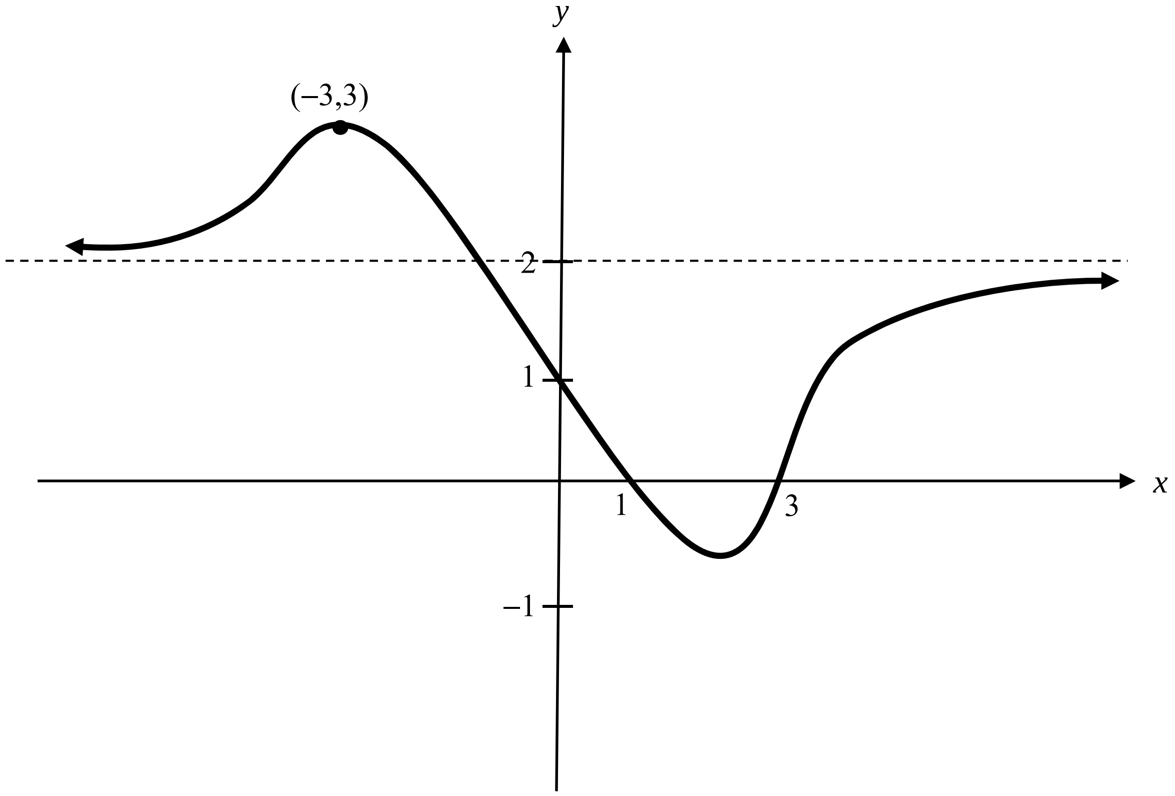
(a) (i) $y = f\left(\frac{x}{2}\right)$



(ii) $y = f\left(|x|\right)$



(iii) $y = \log_2 f(x)$



NSGHS Extension 2 Task 2 2016 – Suggested Solutions and Markers Comments

Section I

1. C

$P(x) = x^3 - 2x^2 - 15x + 36$

$P'(x) = 3x^2 - 4x = 15$

$P'(x) = 0 \Rightarrow (3x+5)(x-3) = 0 \qquad \therefore x = -\frac{5}{3}, 3$

$P(3) = 27 - 18 - 45 + 36 = 0$

2. D

$1-2i$ is also a root, since complex roots occur in conjugate pairs for polynomial with real coefficients.
Sum of roots is 4 $\quad 1+2i+1-2i+\alpha = 4$

$\therefore \alpha = 2$ is another root.

3. A

$(x+1) = A(x-2) + B(x+2)$

sub. $x = 2,\ 3 = 4B \Rightarrow B = \frac{3}{4} \qquad$ sub. $x = -2,\ -1 = -4A \Rightarrow A = \frac{1}{4}$

4. B

$y = \pm\sqrt{f(x)}$

5. D

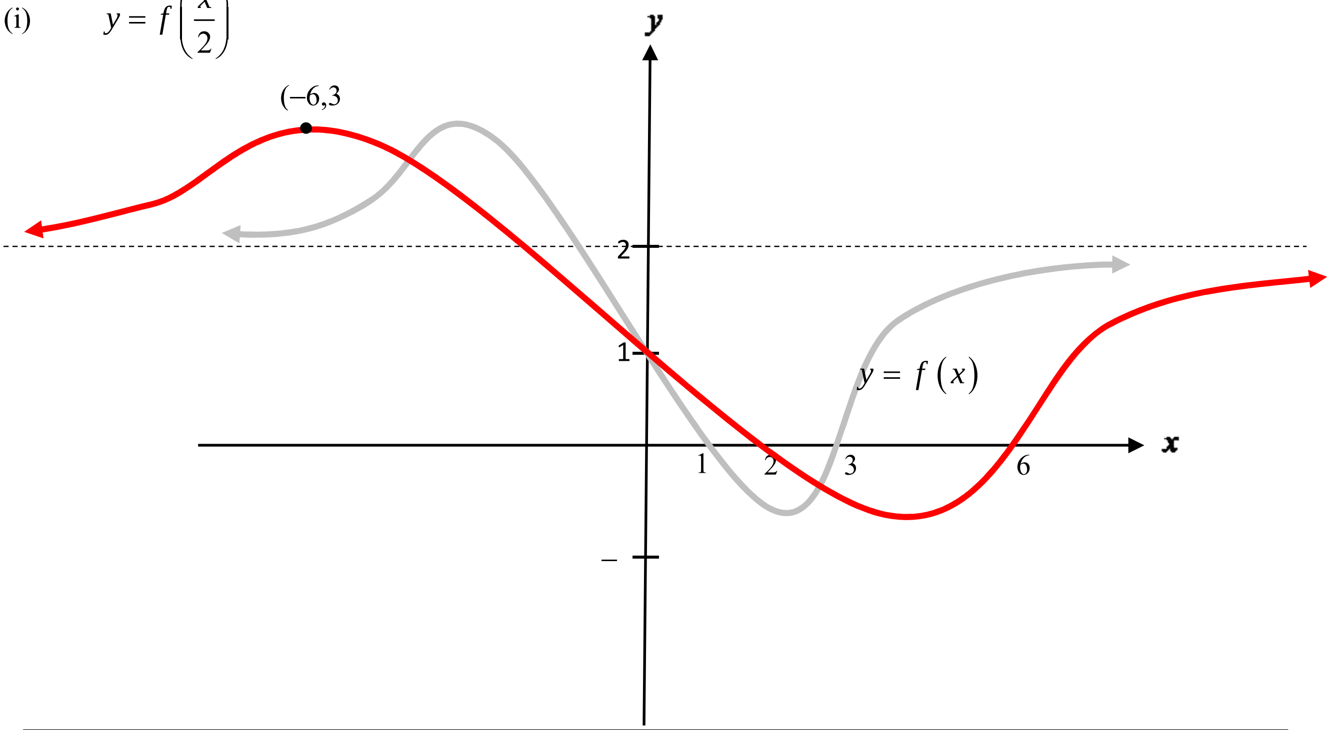
as $x \to \pm\infty,\ f(x) \to \infty.$

as $x \to 0^+,\ f(x) \to \frac{a}{x} = -\infty \qquad$ as $x \to 0^-,\ f(x) \to -\frac{a}{x} = +\infty$

Section II

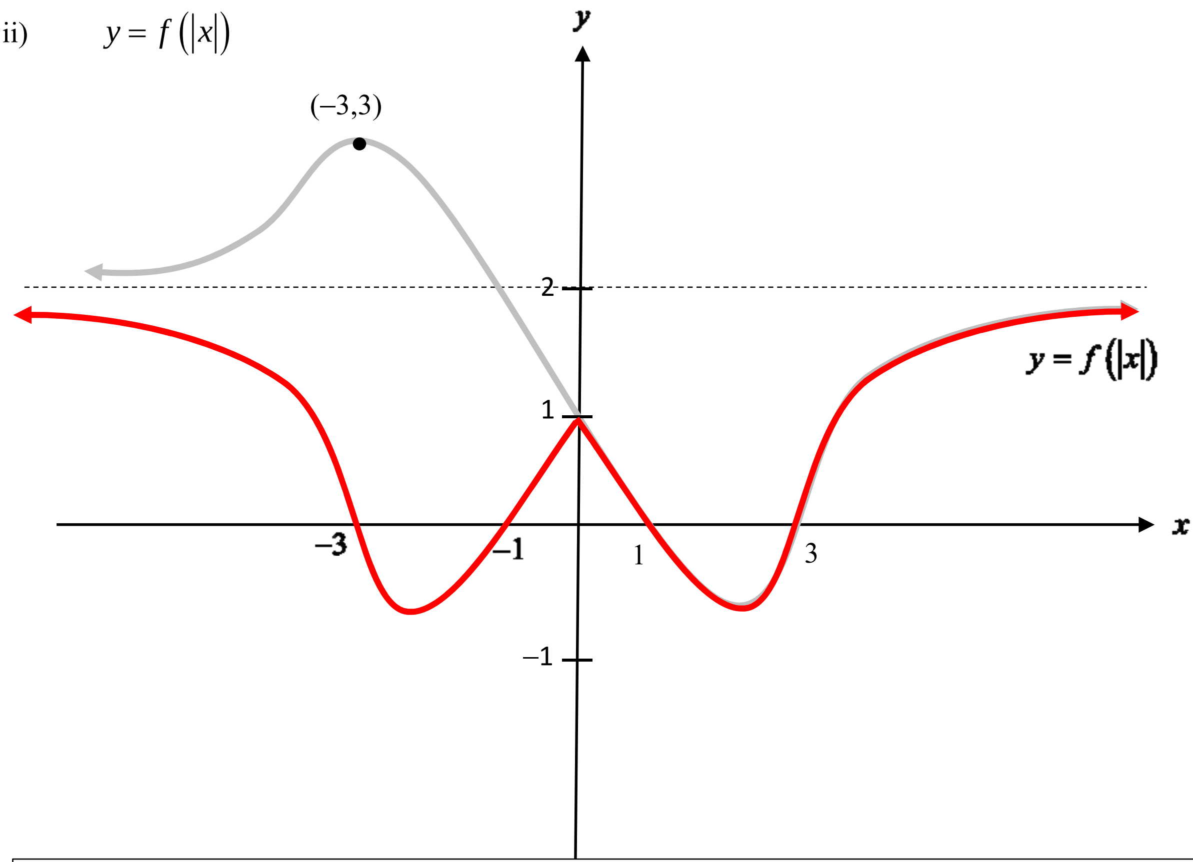
Question 6 (a)

(i) $y = f\left(\frac{x}{2}\right)$



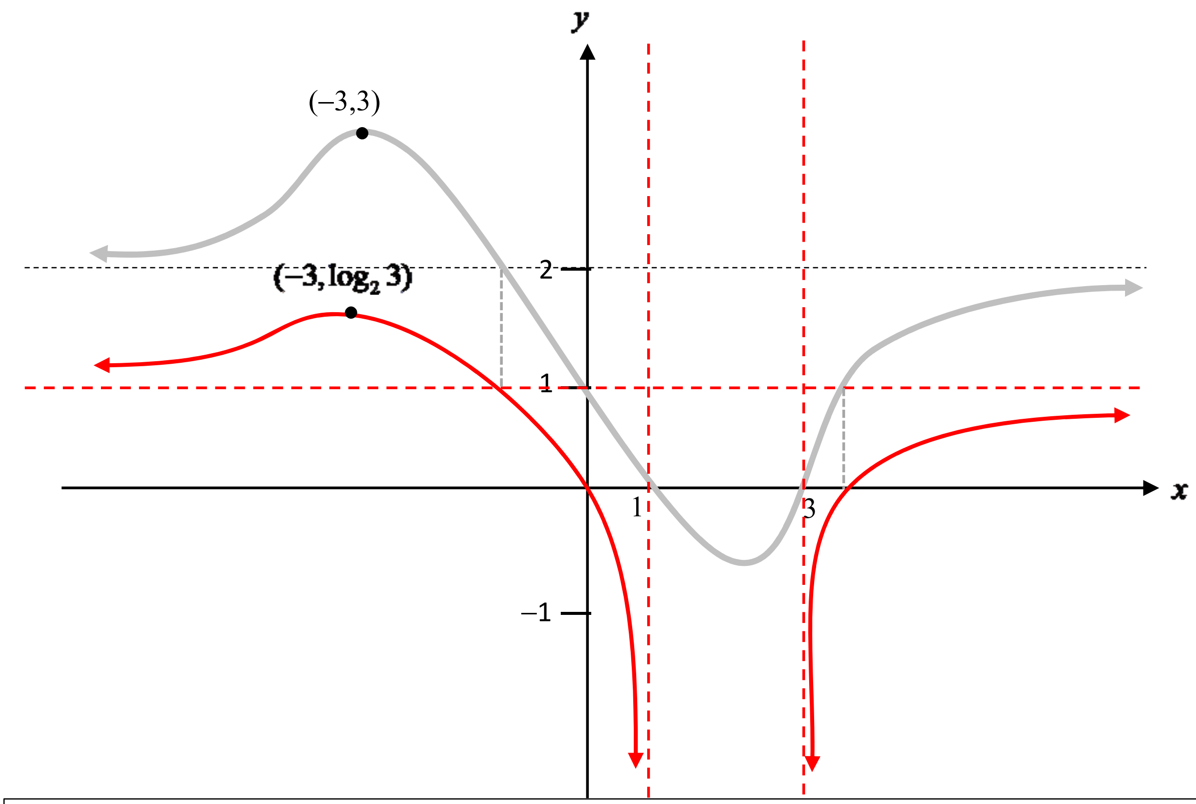
(i) Generally well done. A small number of students drew $y = f(2x)$ or $y = \frac{1}{2} f(x)$

ii) $y = f(|x|)$



(ii)	Most students knew what to do. Remember to label any known intercepts and try to maintain scale and symmetry. Several students marked an intercept of -3 well to the left of the turning point which is at $x = -3$. Fortunately, the marking was lenient with no penalty unless there was a major lapse in scale and/or symmetry.

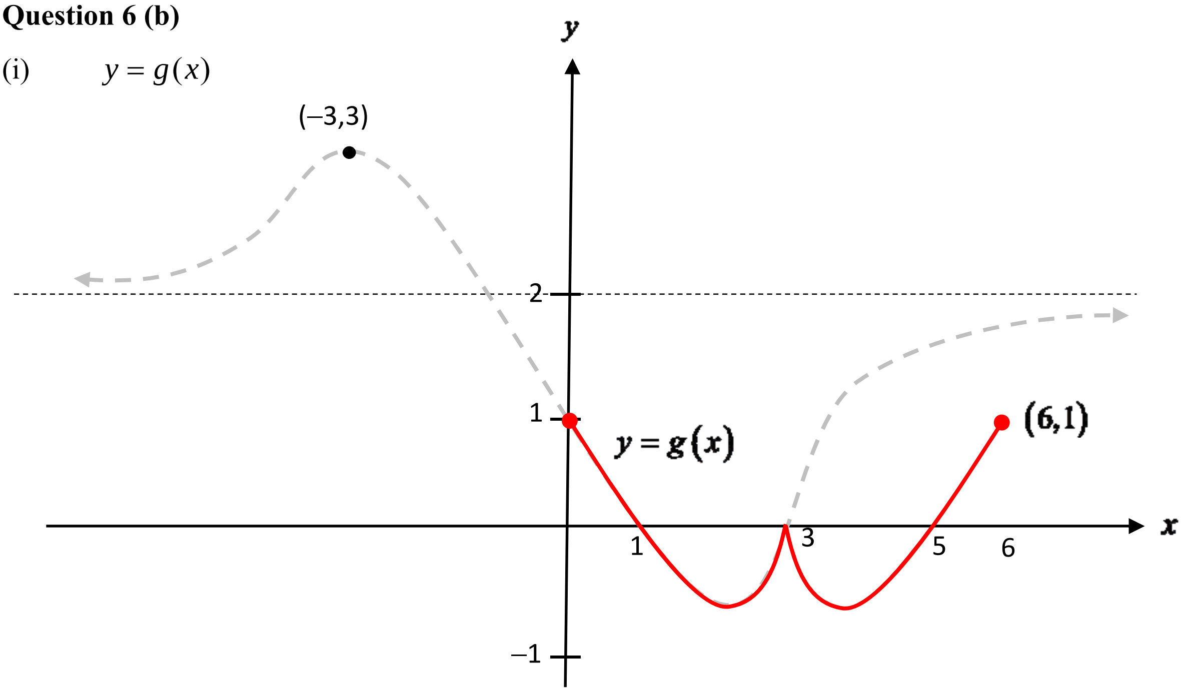
(iii) $y = \log_2 f(x)$



(iii)	Well done overall. Coordinates of the turning point $(-3, \log_2 3)$ are easy to find and should be shown– there was a penalty incurred if this was omitted. The *x*-intercept to the right of $x = 3$ needs to be lined up directly beneath where the curve cuts $y = 1$. This should have been possible especially as you drew in $y = 1$ for the asymptote. There was a penalty if the intercept was shown in the wrong place.

Question 6 (b)

(i) $y = g(x)$



(ii) $g(x) > 0 \Rightarrow \quad 0 \le x < 1, \quad 5 < x \le 6.$

Q6b (i)	This was a piece-meal function. Sketching the piece from $x = 0$ to $x = 3$ earned only ½ a mark as this was basically copying the original graph. The graph of $y = f\left(6 - x\right)$ is a reflection in the line $x = 3$. This should be a known transformation. This section of the graph was not very well done with varying answers reflecting students' uncertainty about this transformation.
(ii)	The solutions are to be written down by observing the graph. One side of the inequality is a strict one and the other side is inclusive. There was a penalty for incorrect inequality signs.

Question 6 (c)

(i) The polynomial with roots $\alpha - 3,\ \beta - 3,\ \gamma - 3$ is given by:

$$Q(x) = (x+3)^3 - (x+3)^2 - 21(x+3) + 45$$
$$= x^3 + 9x^2 + 27x + 27 - (x^2 + 6x + 9) - 21x - 63 + 45$$
$$= x^3 + 8x^2$$

$y = x - 3$

$x = y - 3$

$y = x + 3$

(ii) $Q(x) = 0 \Rightarrow x^2(x+8) = 0$

Roots are $0,\ 0$ and -8.

These roots are $\alpha - 3,\ \beta - 3,\ \gamma - 3$.

$\therefore$ roots of $P(x)$ are $\alpha,\ \beta,\ \gamma$ which are $3,\ 3,\ -5$.

Q6c (i)	Most students knew they needed to replace x with $x+3$. However, many students could not keep their algebra accurate when expanding and simplifying. Work systematically crossing off each term as you simplify. Errors also occurred frequently in expanding $\left(x+3\right)^3$ (!) suggesting that further practice of these basic algebraic skills would be beneficial. Unfortunately errors in this part meant that often part (ii) was not solvable. A small number of students persisted in using sum and product of roots to derive the new polynomial instead of using the Ext 2 technique of transforming roots.
(ii)	This was a "Hence, solve" question. You needed to use part (i). No marks could be earned for using alternate methods.

Question 6 (d)

(i) $3x^2 + 3y^2\dfrac{dy}{dx} - 3(1)y - 3x\dfrac{dy}{dx} = 0$

$$3x^2 - 3y + (3y^2 - 3x)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$= \frac{y - x^2}{y^2 - x}$$

(ii) $\dfrac{dy}{dx} = 0 \Rightarrow y - x^2 = 0 \Rightarrow y = x^2$

sub. $y = x^2$ in (i)

$$x^3 + \left(x^2\right)^3 - 3x\left(x^2\right) = 48$$

$$x^6 - 2x^3 - 48 = 0$$

$$\left(x^3 - 8\right)\left(x^3 + 6\right) = 0$$

$$x = \sqrt[3]{8} = 2, \quad x = \sqrt[3]{-6}$$

6d	(i) Well done. (ii) Generally well done. Most derived the condition for $\dfrac{dy}{dx} = 0$ and substituted back into the original curve. A small number of students could not accurately solve the resulting equation which was reducible to a quadratic. Some students incorrectly discounted the solution $x = \sqrt[3]{-6}$ as they felt it was not real.

Question 7 (a)

(i) Since $1+i$ is a root, $1-i$ is also a zero since $P(x)$ has real coefficients. (conjugate root theorem)

$\left(x-(1+i)\right)\left(x-(1-i)\right)$ is a factor of $P(x)$.

$\left(x^2 - 2x + 2\right)$ is a factor of $P(x)$. $\left[\left(x-\alpha\right)\left(x-\overline{\alpha}\right) = x^2 - 2\left(\operatorname{Re}\alpha\right)x + \left|\alpha\right|^2\right]$

$P(x) = \left(x^2 - 2x + 2\right)\left(x^2 - 6x + 10\right)$ by inspection.

(ii) $P(x) = \left[\left(x^2 - 2x + 1\right) + 1\right]\left[\left(x^2 - 6x + 9\right) + 1\right]$

$$= \left[\left(x-1\right)^2 + 1\right]\left[\left(x-3\right)^2 + 1\right]$$

Now $\left(x-1\right)^2 + 1 > 0$ for $x \in \mathbb{R}$. Similarly, $\left(x-3\right)^2 + 1 > 0$ for $x \in \mathbb{R}$

$\therefore P(x) = \left[\left(x-1\right)^2 + 1\right]\left[\left(x-3\right)^2 + 1\right] > 0$ for $x \in \mathbb{R}$

Alternatively, use the discriminant to show both quadratic factors are positive definite.

7a	(i)	You must mention **real coefficients** in order to use the Conjugate Root Theorem.
	(ii)	Not explained well. Stating that the discriminants of each factor are zero is insufficient. The factors might be **negative** definite – you need to state why they must be **positive** for all x. Some students tried to take cases, without trying to ascertain the values of x for which those cases might apply. Get your terms right. A "positive quartic" is one that is positive for all values of x. People used this term to mean "a quartic whose leading coefficient is positive".

Question 7 (b)

(i) $S_1 = \alpha + \beta + \gamma = 3$ $S_2 = \alpha^2 + \beta^2 + \gamma^2 = 7$ $S_3 = \alpha^3 + \beta^3 + \gamma^3 = 3$

Using relationship between roots and coefficients:

$\alpha + \beta + \gamma = -a \qquad \therefore a = -3$ using S_1

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$$

$$2(\alpha\beta + \beta\gamma + \alpha\gamma) = (\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2)$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{1}{2}\left[(\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2)\right] = \frac{1}{2}\left[(S_1)^2 - S_2\right]$$

$$= \frac{1}{2}(3^2 - 7) = 1$$

Using relationship between roots and coefficients:

$\alpha\beta + \beta\gamma + \alpha\gamma = b \qquad \therefore b = 1$

(ii) Since α, β, γ are roots of $P(x) = 0$

$$\alpha^3 - 3\alpha^2 + \alpha + c = 0$$
$$\beta^3 - 3\beta^2 + \beta + c = 0$$
$$\gamma^3 - 3\gamma^2 + \gamma + c = 0$$

Adding $(\alpha^3 + \beta^3 + \gamma^3) - 3(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta + \gamma) + 3c = 0$

$$3 - 3(7) + (3) + 3c = 0$$
$$3c = 15$$
$$c = 5$$

7b	(i)	Well done
	(ii)	Generally well done. But you can't just quote $\sum\alpha^3 - 3\sum\alpha^2 + \sum\alpha + 3c = 0$. You must derive/explain this.

Question 7 (c)

(i) LHS$= (\cot\theta + i)^n + (\cot\theta - i)^n$

$$= \left(\frac{\cos\theta + i\sin\theta}{\sin\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\sin\theta}\right)^n$$

$$= \frac{\cos n\theta + i\sin n\theta}{\sin^n\theta} + \left(\frac{\cos(-\theta) + i\sin(-\theta)}{\sin\theta}\right)^n$$

using DeMoivre's Theorem and

$\cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta$

$$= \frac{\cos n\theta + i\sin n\theta}{\sin^n\theta} + \frac{\cos(-n\theta) + i\sin(-n\theta)}{\sin^n\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta + \cos n\theta - i\sin(n\theta)}{\sin^n\theta}$$

$$= \frac{2\cos n\theta}{\sin^n\theta} = \text{RHS}$$

(ii) $(x+i)^5+(x-i)^5=0$

Let $x=\cot\theta$

Then $\left(\cot\theta+i\right)^5+\left(\cot\theta-i\right)^5=0$

From (i) $\dfrac{2\cos 5\theta}{\sin^5\theta}=0 \Rightarrow 2\cos 5\theta=0 \quad (\text{provided } \sin\theta=0)$

$5\theta=\pm\dfrac{\pi}{2}+2k\pi \quad k\in\mathbb{Z}$

$\theta=\pm\dfrac{\pi}{10}+\dfrac{2k\pi}{5}$

$\theta=\dfrac{\pi}{10},\dfrac{5\pi}{10},\dfrac{9\pi}{10},\dfrac{13\pi}{10},\dfrac{17\pi}{10}$

Roots of $(x+i)^5+(x-i)^5=0$ correspond to these roots where $x=\cot\theta$.

$x=\cot\dfrac{\pi}{10},\ \cot\left(\dfrac{5\pi}{10}\right),\ \cot\left(\dfrac{9\pi}{10}\right),\ \cot\left(\dfrac{13\pi}{10}\right),\ \cot\left(\dfrac{17\pi}{10}\right)$

$=\cot\dfrac{\pi}{10},\quad 0,\quad -\cot\left(\dfrac{\pi}{10}\right),\ \cot\left(\dfrac{3\pi}{10}\right),\ -\cot\left(\dfrac{3\pi}{10}\right)$ since $\cot\left(\dfrac{5\pi}{10}\right)=0$

(iii) $(x+i)^5+(x-i)^5=$

$x^5(1+1)+5x^4(i+(-i))+10x^3(i^2+(-i)^2)+10x^2(i^3+(-i)^3)+5x(i^4+(-i)^4)+\left(i^5+(-i)^5\right)$

Hence $(x+i)^5+(x-i)^5=2x^5-20x^3+10x$

$2x(x^4-10x^2+5)=0$

$x=0$ or $x^4-10x^2+5=0$

$\therefore x^4-10x^2+5=0$ has roots $\pm\cot\left(\dfrac{\pi}{10}\right),\ \pm\cot\left(\dfrac{3\pi}{10}\right)$.

7c (i) Students who stated $z+\dfrac{1}{z}=2\operatorname{Re}(z)$ without justification could not get full marks – this proof (applied to this problem) was what we were testing for.

(ii) Some people showed that each of these values were roots by substitution. This does not show that they are the **only** roots.

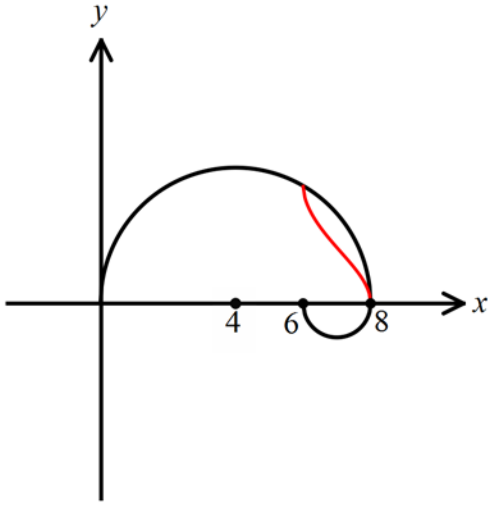
(iii) You had to show sufficient algebra to show where the x^4-10x^2+5 is coming from.

Students who tried to simplify $(x+i)^5+(x-i)^5$ by factorising instead of expanding typically gave up on the algebra and stated the result without proof.

There had to be a serious attempt to explain the loss of the $x=0$ solution.

Question 7 (d)

$$f(x) = \sqrt{-1(x^2 - 8x)} - \sqrt{-1(x^2 - 14x + 48)}$$
$$= \sqrt{-1(x^2 - 8x + (-4)^2 - 16)} - \sqrt{-1(x^2 - 14x + (-7)^2 - 49 + 48)}$$
$$= \sqrt{16 - (x-4)^2} - \sqrt{1 - (x-7)^2}$$



Considering $f(x)$ as the sum of two functions:

$y = \sqrt{16 - (x-4)^2}$ (a semi circle with centre (4,0) and radius 4)

$y = -\sqrt{1 - (x-7)^2}$ (a semi circle with centre (7,0) and radius 1) and then adding ordinates,

The maximum value of $f(x)$ is $f(6) = \sqrt{12} = 2\sqrt{3}$

Alternatively, consider the two parabolas $y = 8x - x^2$ and $y = 14x - 48 - x^2$, graph the square root of both of these functions and then consider the maximum difference between the two.

7d	There had to be some explanation of why $x = 6$ gave the maximum value. A rough graph of the function by addition of ordinates sufficed. Many students did not seem to realise that if a value of x is excluded from the domain of one of the component functions then it is excluded from the domain of the entire function.

End of Solutions